

Capacity-Achieving Rate-Compatible Polar Codes

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Abstract—We present a method of constructing rate-compatible polar codes that are capacity-achieving with low-complexity sequential decoders. The proposed code construction allows for incremental retransmissions at different rates in order to adapt to channel conditions. The main idea of the construction exploits certain common characteristics of polar codes that are optimized for a sequence of degraded channels. The proposed approach allows for an optimized polar code to be used at every transmission thereby achieving capacity. Due to the length limitation of conventional polar codes, the proposed construction can only support a restricted set of rates that is characterized by the size of the kernel when conventional polar codes are used. We thus consider punctured polar codes which provide more flexibility on block length by controlling a puncturing fraction. We show the existence of capacity-achieving punctured polar codes for any given puncturing fraction. Using punctured polar codes as constituent codes, we show that the proposed rate-compatible polar code is capacity-achieving for an arbitrary sequence of rates and for any class of degraded channels.

Index Terms—Polar codes, channel capacity, capacity-achieving codes, rate-compatibility, retransmissions, HARQ-IR.

I. INTRODUCTION

Polar codes, proposed by Arikan [1], achieve the symmetric capacity of the binary-input discrete memoryless channels using a low-complexity successive cancellation (SC) decoder. The finite-length performance of polar codes can be improved by deploying list decoder enabling polar codes to approach the performance of optimal maximum-likelihood (ML) decoder [2]. Furthermore, a polar code concatenated with a simple CRC outperforms well-optimized LDPC and Turbo codes even for short block lengths [2]. Due to their good performance and low complexity, polar codes are being considered for possible use in future wireless communication systems (e.g., 5G).

Wireless broadband systems operate in the presence of time-varying channels and therefore require flexible and adaptive transmission techniques. For such systems, hybrid automatic repeat request based on incremental redundancy (HARQ-IR) schemes are often used, where parity bits are sent in an incremental fashion depending on the quality of the time-varying channel. In HARQ-IR scheme, a number of parity bits chosen according to a rate requirement, are sent by the transmitter. IR systems require the use of *rate-compatible* codes typically obtained by puncturing. For a code to be rate-compatible, the set of parity bits of a higher rate code should be a subset of the set of parity bits of a lower rate code. This allows the receiver that fails to decode at a particular rate, to

request only additional parity bits from the transmitter. For this reason, there has been extensive research on the construction of rate-compatible Turbo codes and LDPC codes (see [3], [4] and references therein).

Although polar codes can achieve the capacity of symmetric binary-input channel, their rate-compatible constructions are not in general capacity-achieving. Determining puncturing patterns that result in good performance was considered in [5], [6], [7], [8]. More recently, an efficient algorithm for joint optimization of puncturing patterns and the set of information bits of the code was proposed and shown to outperform LDPC codes [9]. Because in [5], [7], [8], [9] an information set is optimized according to a puncturing pattern, these methods cannot be used to design a family of rate-compatible punctured codes as required for HARQ-IR, where the same information set (generally optimized for the mother code) should be used for all punctured codes in the family. In [6], a heuristic search algorithm was presented to design a good puncturing pattern for a fixed information set. However, finding an optimal rate-compatible puncturing pattern with low complexity is still an open problem.

In this paper, we present a family of rate-compatible polar codes that are symmetric-capacity achieving. The main idea of our construction exploits certain common characteristics of polar codes that are optimized for a sequence of degraded channels. In our approach, we construct a concatenated polar code that is decoded by a sequence of parallel polar decoders. We refer to this construction as *parallel concatenated polar codes*. The proposed code construction allows for incremental retransmissions at different transmission rates and can therefore be used for HARQ-IR. Furthermore, an optimized polar code is used at every retransmission thereby achieving symmetric capacity, for any class of degraded channels. A similar idea was proposed by Li et. al. in [10]. Due to the length limitation of polar codes, our proposed construction of rate-compatible polar codes can achieve the capacity only for a sequence of rates that satisfy a certain relationship (as specified in Theorem 3). In order to support any arbitrary sequence of rates, we present capacity-achieving punctured polar codes which can provide more flexibility on block length by controlling a puncturing fraction. Using such punctured polar codes, we show that the proposed rate-compatible polar code is capacity-achieving for an arbitrary sequence of rates.

II. PRELIMINARIES

In this section, we provide some basic definitions and background that will be used in the sequel.

A. Rate-Compatible Codes

We start with a definition of rate-compatible codes. To simplify notation, let $[K] \triangleq \{1, 2, \dots, K\}$ for any positive integer K and let $a^n = (a_1, \dots, a_n)$ denote a vector of length n . Given a fixed number of information bits k , a family of codes, or code family, $\mathcal{C} = \{\mathcal{C}_1^{\bar{n}_1}, \mathcal{C}_2^{\bar{n}_2}, \dots, \mathcal{C}_K^{\bar{n}_K}\}$ with respective block lengths $\bar{n}_1 < \bar{n}_2 < \dots < \bar{n}_K$ and corresponding rates $R_1 > R_2 > \dots > R_K$, where $R_i = k/\bar{n}_i$, is said to be *rate-compatible* if there exist a sequence of encoding functions $\{\bar{e}_i(\cdot)\}_{i \in [K]}$, where $\bar{e}_i : \{0, 1\}^k \rightarrow \{0, 1\}^{\bar{n}_i}$ is the corresponding encoding function of \mathcal{C}_i for $i \in [K]$, and a sequence of projection operators $\{\pi_i(\cdot)\}_{i \in [K-1]}$, where $\pi_i : \{0, 1\}^{\bar{n}_{i+1}} \rightarrow \{0, 1\}^{\bar{n}_i}$ simply takes \bar{n}_i of the \bar{n}_{i+1} coordinates of its input as output, such that for each $i \in [K-1]$,

$$\bar{e}_i(u^k) = \pi_i(\bar{e}_{i+1}(u^k)), \quad (1)$$

for every possible information block $u^k \in \{0, 1\}^k$. We refer to such a sequence of encoding functions as a sequence of *nested encoding functions*. Condition (1) assures that the set of parity bits of a higher rate code is a subset of the set of parity bits of a lower rate code and therefore the code can be used for HARQ-IR.

It follows immediately that a family of *linear* codes $\mathcal{C} = \{\mathcal{C}_1^{\bar{n}_1}, \mathcal{C}_2^{\bar{n}_2}, \dots, \mathcal{C}_K^{\bar{n}_K}\}$ is rate-compatible if and only if there exists a sequence of generator matrices $\{\mathbf{G}_i\}_{i \in [K]}$, each corresponding to a member in \mathcal{C} , such that the columns of \mathbf{G}_i is a subset of those of \mathbf{G}_{i+1} for every $i \in [K-1]$. Accordingly, we refer to $\{\mathbf{G}_i\}_{i \in [K]}$ as a sequence of *nested generating matrices*.

Let $W_1 \succeq W_2 \succeq \dots \succeq W_K$ denote a sequence of successively degraded discrete memoryless channels (DMC) with a common input alphabet \mathcal{X} , respective output alphabets $\{\mathcal{Y}_i\}_{i \in [K]}$, respective transition probability distributions $\{W_i(y|x)\}_{i \in [K]}$ where $x \in \mathcal{X}$ and $y \in \mathcal{Y}_i$, and respective capacities $I(W_1) > I(W_2) > \dots > I(W_K)$. A sequence of rate-compatible code families, $\mathcal{C}_m = \{\mathcal{C}_1^{\bar{n}_{1,m}}, \mathcal{C}_2^{\bar{n}_{2,m}}, \dots, \mathcal{C}_K^{\bar{n}_{K,m}}\}$ for $m \in \mathbb{N}$, designed for a monotonically increasing sequence of information block sizes $\{k_m\}_{m \in \mathbb{N}}$, is said to be *capacity-achieving with respect to* $\{W_i\}_{i \in [K]}$ if, for every m , there exist a sequence of decoding functions $\{d_{i,m}(\cdot)\}_{i \in [K]}$, where $d_{i,m} : \mathcal{Y}^{\bar{n}_{i,m}} \rightarrow \{0, 1\}^{k_m}$, and a corresponding sequence of nested encoding functions $\{e_{i,m}(\cdot)\}_{i \in [K]}$ such that for any $\epsilon > 0$, we have, for every $i \in [K]$,

$$R_{i,m} \triangleq k_m/\bar{n}_{i,m} > I(W_i) - \epsilon, \quad \text{and} \quad (2)$$

$$\Pr(u^{k_m} \neq d_{i,m}(y^{\bar{n}_{i,m}})) < \epsilon, \quad (3)$$

for the joint probability distribution given by $p(u^{k_m}, y^{\bar{n}_{i,m}}) = 2^{-k_m} W_i^{\bar{n}_{i,m}}(y^{\bar{n}_{i,m}} | e_i(u^{k_m}))$, where $W_i^n(y^n | x^n) \triangleq \prod_{l=1}^n W_i(y_l | x_l)$, for all sufficiently large m .

In this paper, we focus on binary-input discrete memoryless channels (B-DMC) with input alphabet $\mathcal{X} = \{0, 1\}$ and any

output alphabet \mathcal{Y} . We denote the transition probabilities of any B-DMC W by $W(y|0)$ and $W(y|1)$ for all $y \in \mathcal{Y}$.

B. Polar Codes

Let $2^{\mathbb{N}}$ denote the set of powers of two. For any $n \in 2^{\mathbb{N}}$, let $\mathbf{P}_n \triangleq \mathbf{P}_2^{\otimes \log(n)}$ denote the rate-one generator matrix of all polar codes with block length n , where \mathbf{P}_2 is the 2-by-2 Arikan kernel [1]. As shown in [1], as n increases, under successive cancellation (SC) decoding, a fraction of the rows of \mathbf{P}_n leads to good bit-channels suitable for carrying information bits while the rest forms bad bit-channels whose input should be frozen to known values, assumed to be zero in this paper. Thus a polar code of length n is completely determined by the rate-one generator matrix \mathbf{P}_n and the information set \mathcal{A} that specifies the set of good bit-channel locations, with the rate of the code given by the ratio $R = |\mathcal{A}|/n$. Let $\mathcal{C}(n, R, \mathcal{A})$ denote a polar code of rate R with information set \mathcal{A} . Also let $\mathbf{P}_n^{\mathcal{A}}$ denote its $|\mathcal{A}| \times n$ generator matrix formed by only taking the rows of \mathbf{P}_n that correspond to \mathcal{A} . The specific orderings of rows within $\mathbf{P}_n^{\mathcal{A}}$ is unimportant in the following discussion.

Given a finite sequence of channels, $\{W_i\}_{i \in [K]}$, a sequence of polar codes $\{\mathcal{C}(n, R_i, \mathcal{A}_i)\}_{i \in [K]}$ of a common block length n with different rates $R_1 > R_2 > \dots > R_K$, each corresponding to one channel, can be obtained by selecting a different information set \mathcal{A}_i for each channel W_i . We refer to a sequence of polar codes $\{\mathcal{C}(n, R_i, \mathcal{A}_i)\}_{i \in [K]}$ having the same block length n as a sequence of *nested polar codes* if their respective information sets are nested, i.e.

$$\mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \dots \supseteq \mathcal{A}_K. \quad (4)$$

For given nested information sets $\mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \dots \supseteq \mathcal{A}_K$, we define an $|\mathcal{A}_1| \times n$ generator matrix $\mathbf{P}_n^{(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_K)}$ with a partial ordering of rows according to $\{\mathcal{A}_i\}$ as

$$\mathbf{P}_n^{(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_K)} = \begin{bmatrix} \mathbf{P}_n^{\mathcal{A}_K} \\ \mathbf{P}_n^{\mathcal{A}_{K-1} \setminus \mathcal{A}_K} \\ \vdots \\ \mathbf{P}_n^{\mathcal{A}_1 \setminus \mathcal{A}_2} \end{bmatrix}. \quad (5)$$

Note that the only difference between $\mathbf{P}_n^{\mathcal{A}_1}$ and $\mathbf{P}_n^{(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_K)}$ is that the ordering of rows is more specifically defined in the latter. For any index set $\mathcal{D} \subseteq [n]$, let $\mathbf{P}_n^{(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_K)}(\mathcal{D})$ denote a submatrix of $\mathbf{P}_n^{(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_K)}$ consisting of rows whose indices belong to \mathcal{D} .

Given a channel W_i , $i \in [K]$, we define the information set for a fixed ϵ as

$$\mathcal{A}_{i,n}(\epsilon) = \{j \in [n] : Z(W_{i,n}^{(j)}) \leq \epsilon\}, \quad (6)$$

where $W_{i,n}^{(j)}$ denotes the j -th bit-channel resulted from a polar code with block length n applied to W_i , and $Z(W)$ denotes the Bhattacharyya parameter of bit-channel W given by $Z(W) \triangleq \sum_{y \in \mathcal{Y}} \sqrt{W(y|0)W(y|1)}$. The following result follows directly from [11, Lemma 4.7].

Lemma 1: [11, Lemma 4.7] Given a sequence of successively degraded channels $W_1 \succeq W_2 \succeq \dots \succeq W_K$,

$\{\mathcal{C}(n, R_{i,n}, \mathcal{A}_{i,n}(\epsilon))\}_{i \in [K]}$ is a sequence of nested polar codes for any ϵ .

The set of good bit-channels can alternatively be defined in terms of the symmetric capacity $I(W)$. When clear from the context, the dependency of R_i, \mathcal{A}_i on ϵ will be omitted.

Relation (4) is a key property that we will use for constructing rate-compatible polar codes. However, since nested polar codes have the same block length and varying information block sizes, they cannot directly be used for HARQ-IR, which in contrast assumes a fixed information block size and allows varying block lengths to achieve different rates with a given error probability tolerance. To obtain a family of (capacity-achieving) rate-compatible codes, we need to construct multi-sequences of nested polar codes as described later.

III. PUNCTURED POLAR CODES

Polar codes with arbitrary lengths can be obtained by puncturing. A polar code of length n is punctured by removing a set of s columns from its generator matrix, which has the effect of reducing the codeword length from n to $n-s$. We let $\alpha = s/n$ denote a *puncturing fraction*. Formally, a punctured polar code of post-puncturing block length n is characterized by its "mother" (unpunctured) polar code of block length $n_u \in 2^{\mathbb{N}}$ and a puncturing pattern $p^{n_u} = (p_1, p_2, \dots, p_{n_u}) \in \{0, 1\}^{n_u}$ with $p_i = 0$ indicating that the i th coded bit is punctured and thus not transmitted. For a given p^{n_u} , let $\pi_{p^{n_u}} : \mathcal{Y}^{n_u} \rightarrow \mathcal{Y}^n$ be a projection operator that copies $n = w_H(p^{n_u})$ coordinates of its input as its output based on the puncturing pattern specified by p^{n_u} , where $w_H(p^{n_u})$ denotes the number of ones in p^{n_u} , i.e. $y^n = \pi_{p^{n_u}}(y^{n_u})$ containing the coordinates of y^{n_u} corresponding to the locations of ones in p^{n_u} . The notion of bit-channels in conventional polar codes can be extended to punctured polar codes in a straightforward manner as follows. For a given (unpunctured) polar code of block length $n_u \in 2^{\mathbb{N}}$ and puncturing pattern p^{n_u} , we define the transition probability of the i -th bit channel of the corresponding punctured polar code as

$$\begin{aligned} W^{(i)}(y^n, u^{i-1}, p^{n_u} | u_i) \\ = \frac{1}{2^{n_u-1}} \sum_{u_{i+1}^{n_u}} \sum_{y^{n_u} \in \pi_{p^{n_u}}^{-1}(\{y^n\})} W^{n_u}(y^{n_u} | u^{n_u} \mathbf{P}_{n_u}), \end{aligned} \quad (7)$$

where $W^{n_u}(y^{n_u} | x^{n_u}) = \prod_{j \in [n_u]} W(y_j | x_j)$ and $\pi_{p^{n_u}}^{-1}(S) \triangleq \{y^{n_u} \in \mathcal{Y}^{n_u} : \pi_{p^{n_u}}(y^{n_u}) \in S\}$ denotes the inverse image of $\pi_{p^{n_u}}(\cdot)$. The information set of a punctured polar code can be defined in a similar manner as in (6).

Let $\mathcal{C}(n, R, \mathcal{A}, p^{n_u})$ denote a punctured polar code of (post-puncturing) block length n , rate R , information set \mathcal{A} , and a puncturing pattern p^{n_u} . Similarly, one can let $\mathbf{P}_{n,p^{n_u}}$ denote the matrix obtained by removing the columns of \mathbf{P}_{n_u} according to the locations of zeros in the puncturing pattern p^{n_u} , and for any nested information sets $\mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \dots \supseteq \mathcal{A}_K$, one can define $\mathbf{P}_{n,p^{n_u}}^{(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_K)}$ in the same manner as in (5). However, in most cases of our discussion below, there is only one puncturing pattern for each block length n , and therefore we will omit p^{n_u} and use same notations and similar

terminologies for both punctured and unpunctured polar codes whenever it is clear from context. In particular, we refer to a family of rate-compatible codes $\mathcal{C} = \{\mathcal{C}_1^{\bar{n}_1}, \mathcal{C}_2^{\bar{n}_2}, \dots, \mathcal{C}_K^{\bar{n}_K}\}$ as a family of *rate-compatible polar codes* if a (punctured) polar code is used in every transmission.

In the proposed scheme that will be explained in Section IV, any good punctured (or shortened) polar code can be used for length flexibility and it is left for a future work to design a good punctured polar code for short block lengths. In order to show that the proposed rate-compatible polar codes achieve the capacity, we will use a capacity-achieving punctured polar code for any desired puncturing fraction, whose existence is shown in the following theorem.

Theorem 1: Consider any B-DMC W with $I(W) > 0$. For any fixed $R < I(W)$, $\beta < \frac{1}{2}$, and puncturing fraction $\alpha \in (0, 1)$, there exists a sequence of punctured polar codes, each with respective block length $n = \lfloor (1-\alpha)2^m \rfloor$ and associated information sets $\mathcal{A}_m \subset [2^m]$, $m \in \mathbb{N}$, such that $|\mathcal{A}_m| \geq \lfloor 2^m(1-\alpha) \rfloor R = nR$ and

$$P_{e,j,m} \leq O(2^{-2^{m\beta}}) = O(2^{-n^\beta}), \quad (8)$$

for all $j \in \mathcal{A}_m$ and for all $m \in \mathbb{N}$, where $P_{e,j,m}$ denote the error probability of j -th polarized bit channel of the m -th punctured polar code.

Proof: See the long version of this paper [12]. ■

IV. MAIN RESULTS

We state two theorems that are the main results of this paper.

Theorem 2: For any sequence of successively degraded channels $W_1 \succeq W_2 \succeq \dots \succeq W_K$, there exists a sequence of rate-compatible polar code families that is capacity-achieving.

Proof: See the long version of this paper [12]. ■

In case puncturing is not used, the constraint on a polar code block length $n = 2^l$, $l \in \mathbb{N}$ reduces the set of rates that can be supported by the proposed coding scheme. In particular, we have the following:

Theorem 3: For any sequence of successively degraded channels $W_1 \succeq W_2 \succeq \dots \succeq W_K$ with corresponding symmetric capacities $I(W_1) > I(W_2) > \dots > I(W_K) > 0$, there exists a sequence of rate-compatible (non-punctured) polar code families that is capacity-achieving if and only if, for each $i \in \{2, \dots, K\}$,

$$I(W_i) = \frac{I(W_1)}{1 + \sum_{j=2}^i 2^{\ell_j}}, \quad (9)$$

for some $\ell_j \in \mathbb{Z}$.

Proof: See the long version of this paper [12]. ■

Below, we explain our main idea for the construction of a family of rate-compatible polar codes. A general code construction will be explained in Section V.

To transmit k information bits over K channels $W_1 \succeq W_2 \succeq \dots \succeq W_K$ at rates $R_1 > \dots > R_K$, we generate K (punctured) polar codes $\mathcal{C}(n_i, R_i, \mathcal{A}_i^{(i)})$, where n_i is the block length of the i -th transmission and is chosen such that

$$R_i = \frac{k}{\sum_{j=1}^i n_j} = \frac{k}{\bar{n}_i}, \quad (10)$$

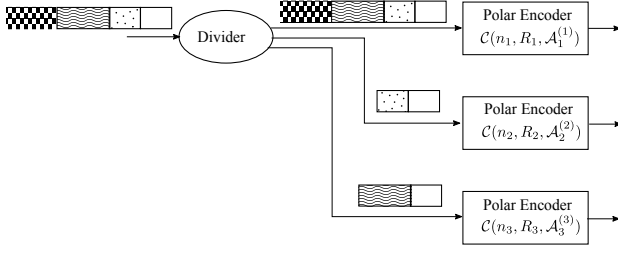


Fig. 1. Encoder structure of rate-compatible polar codes for $K = 3$.

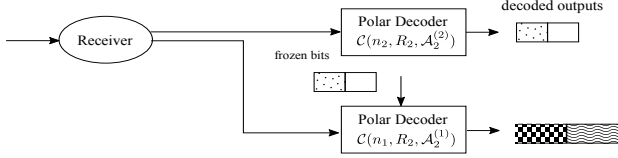


Fig. 2. Sequential decoder of rate-compatible polar codes for rate R_2 .

where $\bar{n}_i \triangleq \sum_{j=1}^i n_j$ is the effective block length after i transmissions. In addition, for each block length n_i , $i = 1, \dots, K$, we construct a sequence of nested (punctured) polar codes $\{\mathcal{C}(n_i, R_j, \mathcal{A}_j^{(i)})\}_{j=i}^K$ with rates $\{R_i, \dots, R_K\}$ such that¹

$$|\mathcal{A}_j^{(i)}| = n_i R_j, \quad (11)$$

for $j \in [K]$. Exact choice of information sets $\mathcal{A}_j^{(i)}$ is described in Sec. V. We let $\mathcal{I}^{(i)} \triangleq \bigcup_{j=1}^{i-1} \mathcal{I}_{i-1}^{(j)}$ where $\mathcal{I}^{(1)} = [k]$ and $\mathcal{I}_{i-1}^{(j)}$ is the index set of information bits that are used to convert the polar code in each of previous transmissions from rate R_{i-1} (that cannot be supported by the channel) to corresponding codes of rate R_i , namely, $\mathcal{I}_{i-1}^{(j)}$ contains the indices of information bits corresponding to $\mathcal{A}_{i-1}^{(j)} \setminus \mathcal{A}_i^{(j)}$. At transmission i , we then use the code $\mathcal{C}(n_i, R_i, \mathcal{A}_i^{(i)})$ to transmit some part of information bits $\mathcal{I}^{(i)}$ as shown in Fig. 1. This is possible because $\mathcal{I}^{(i)}$ and $\mathcal{A}_i^{(i)}$ are of the same size, that is:

$$\begin{aligned} |\mathcal{I}^{(i)}| &= \sum_{j=1}^{i-1} |\mathcal{A}_{i-1}^{(j)}| - |\mathcal{A}_i^{(j)}| \stackrel{(a)}{=} \sum_{j=1}^{i-1} q_{i-1}^{(j)} = k - R_i \sum_{j=1}^{i-1} n_j \\ &= k - \frac{k \sum_{j=1}^{i-1} n_j}{\sum_{j=1}^i n_j} = n_i R_i \stackrel{(b)}{=} |\mathcal{A}_i^{(i)}|, \end{aligned} \quad (12)$$

where $q_i^{(j)} \triangleq n_j(R_i - R_{i+1})$ for each $i \in [K-1]$ and (a) and (b) follow by (11).

Code $\mathcal{C}(n_i, R_i, \mathcal{A}_i^{(i)})$ is the code of the highest rate in the sequence of common block length n_i . Each such sequence satisfies the property (4) which will be exploited in decoding. In particular, suppose that m retransmissions, where $m \in [K]$, are needed and hence rate $I(W_m)$ is the highest rate that can be supported by the channel. The decoder starts by decoding the information bits of the polar code $\mathcal{C}(n_m, R_m, \mathcal{A}_m^{(m)})$. It

¹For the ease of exposition, it is assumed that every n_i is sufficiently large such that $n_i R_j$ is an integer. Note that using floor function to make every $n_i R_j$ an integer does not change the main results of this paper.

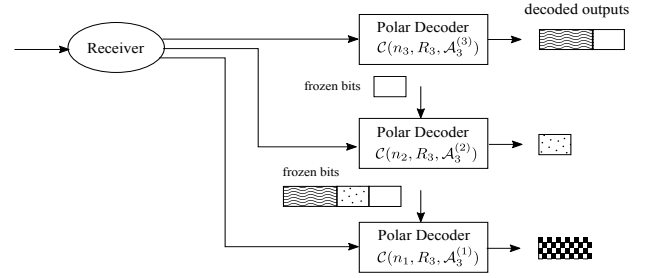


Fig. 3. Sequential decoder of rate-compatible polar codes for rate R_3 .

then uses some of these decoded bits as frozen bits in the polar code $\mathcal{C}(n_{m-1}, R_{m-1}, \mathcal{A}_{m-1}^{(m-1)})$ thereby, due to property (4), turning this code into a polar code $\mathcal{C}(n_{m-1}, R_m, \mathcal{A}_{m-1}^{(m-1)})$ in the same sequence (i.e., of the same block length n_{m-1}) but of lower rate R_m which is supported by the channel. Hence, the information bits of this obtained code can be decoded. It then repeats this sequential decoding over m stages as shown in Figs. 2 and 3, where in each stage, it decodes additional information bits using a polar code of rate R_m . As we show later in Theorem 2, the chosen transmit rates R_i as defined in (10) will approach the corresponding channel capacity $I(W_i)$, as n_i increases for all $i = 1, 2, \dots, K$.

V. GENERAL CODE CONSTRUCTION

In this section, we describe a general method of constructing rate compatible (punctured) polar codes through concatenation of generating matrices of multiple polar codes with (possibly) different block lengths. We refer to the resulting class of codes as *parallel concatenated polar (PCP) codes*.

A. Parallel Concatenated Polar (PCP) Codes

Formally, given an information block size k and a set of subblock lengths $\{n_i\}_{i \in [K]}$, a K -level PCP code of overall block length \bar{n}_K and rate R_K is characterized by a collection of K sequences of nested polar codes $\{\mathcal{C}(n_i, R_j, \mathcal{A}_j^{(i)})\}_{j \geq i, i \in [K]}$ and a collection of K bit-mappings, $h^{(i)}: [|\mathcal{A}_i^{(i)}|] \rightarrow [k]$ for $i \in [K]$, that satisfy the following conditions: (i) $R_i = k / \sum_{j=1}^i n_j$; (ii) $\mathcal{A}_i^{(i)} \supseteq \mathcal{A}_{i+1}^{(i)} \supseteq \dots \supseteq \mathcal{A}_K^{(i)}$ for each $i \in [K]$; (iii) $|\mathcal{A}_j^{(i)}| = n_i R_j$ for all $j \geq i$ and $i \in [K]$.

Note that a total of $K(K+1)/2$ (punctured) polar codes, covering different information-set sizes $|\mathcal{A}_j^{(i)}|$ for all $j \geq i$ and $i \in [K]$, is involved in a K -level PCP code. The $k \times R_K$ generator matrix of a K -level PCP code is fully determined by $(\{\mathcal{C}(n_i, R_j, \mathcal{A}_j^{(i)})\}_{j \geq i, i \in [K]}, \{h^{(i)}\}_{i \in [K]})$ and is a concatenation of submatrices of $\{\mathbf{P}_{n_i}\}_{i \in [K]}$ in the form $\mathbf{G}_K = [\mathbf{S}_1 \mathbf{S}_2 \dots \mathbf{S}_K]$, where \mathbf{S}_i is a $k \times n_i$ matrix whose non-zero rows come from the rows of \mathbf{P}_{n_i} and are indexed by the set $\mathcal{I}^{(i)} = h^{(i)}([|\mathcal{A}_i^{(i)}|])$. More precisely, the m -th row of the matrix $\mathbf{P}_{n_i}^{(\mathcal{A}_i^{(i)}, \mathcal{A}_{i+1}^{(i)}, \dots, \mathcal{A}_K^{(i)})}$, which is a row-permuted submatrix of \mathbf{P}_{n_i} , is placed at the $h^{(i)}(m)$ -th row of \mathbf{S}_i , for every $m \in [|\mathcal{A}_i^{(i)}|]$, while all other $(k - n_i R_i)$ rows of \mathbf{S}_i are zero, where $|\mathcal{A}_i^{(i)}| = n_i R_i$. The matrix \mathbf{S}_i defines the “ i -th level” of the PCP code.

A key feature of a PCP code is that it is sequentially decodable level-by-level if the bit-mapping functions $\{h^{(i)}(\cdot)\}_{i \in [K]}$ are properly related in accordance with $\{\mathcal{A}_j^{(i)}\}$. We define their proper relationships in a recursive manner in the following. Suppose that we are given matrices $\{\mathbf{P}_{n_j}^{(\mathcal{A}_j^{(j)}, \mathcal{A}_{j+1}^{(j)}, \dots, \mathcal{A}_K^{(j)})}\}_{j \in [K]}$ as defined in (5) and that we have already determined submatrices $\{\mathbf{S}_j\}_{j=1}^i$ of \mathbf{G}_K up to the i -th level for some $i \geq 1$. Also suppose that we know the corresponding bit mapping $h^{(j)} : [n_j R_j] \rightarrow [k]$ that specifies the non-zero rows of \mathbf{S}_j that contains the rows of $\mathbf{P}_{n_j}^{(\mathcal{A}_j^{(j)}, \mathcal{A}_{j+1}^{(j)}, \dots, \mathcal{A}_K^{(j)})}$ for all $j \in [i]$, with $h^{(1)}(\cdot)$ simply defined as $h^{(1)}(m) = m$ for $m \in [k]$. We now define $h^{(i+1)}(\cdot)$ in terms of $\{h^{(m)}(\cdot)\}_{m \in [i]}$. To simplify notation, we define $Q_0^{(i+1)} \triangleq 0$ and for each $j \in [i]$, $Q_j^{(i+1)} = \sum_{l=1}^j q_l^{(l)}$.

Now we partition the domain of $h^{(i+1)}(\cdot)$, namely $[n_{i+1} R_{i+1}]$ into disjoint sets as $[n_{i+1} R_{i+1}] = \bigcup_{j=1}^i \mathcal{J}_j^{(i+1)}$, where $\mathcal{J}_j^{(i+1)} \triangleq \{Q_{j-1}^{(i+1)} + 1, Q_{j-1}^{(i+1)} + 2, \dots, Q_{j-1}^{(i+1)} + q_j^{(j)}\}$ is a set of consecutive integers for all $j \in [i]$. We can now define a bit-mapping $h^{(i+1)} : [n_{i+1} R_{i+1}] \rightarrow \mathcal{I}^{(i+1)} \subseteq [k]$ for S_{i+1} in a piece-wise fashion as

$$h^{(i+1)}(m) = h^{(j_m)}(m - Q_{j_m-1}^{(i+1)} + n_{j_m} r_{j_m} - \sum_{l=j_m}^i q_l^{(j_m)}),$$

for $m \in [n_{i+1} R_{i+1}]$, where j_m denotes the index of the interval $\mathcal{J}_{j_m}^{(i+1)}$ that contains integer m . Note that the image $\mathcal{I}_i^{(j)} \triangleq h^{(i+1)}(\mathcal{J}_j^{(i+1)})$ on the set $\mathcal{J}_j^{(i+1)}$ is the index set of information bits that needs to be decoded, and subsequently frozen, in order to convert the polar code $\mathcal{C}(n_j, R_i, \mathcal{A}_i^{(j)})$ of rate R_i to the polar code $\mathcal{C}(n_j, R_{i+1}, \mathcal{A}_{i+1}^{(j)})$ for all $j \in [i]$ and $i \in [K-1]$ within the respective nested polar code sequences. With this definition of $\{h^{(i)}(\cdot)\}_{i \in [K]}$, it can be easily shown that a K -level PCP code can be decoded sequentially from the polar code at level K back to level 1. A K -level PCP code $(\{\mathcal{C}(n_i, R_j, \mathcal{A}_j^{(i)}\}_{j \geq i, i \in [K]}, \{h^{(i)}\}_{i \in [K]})$ clearly induces a family of rate-compatible linear codes $\mathcal{C} = \{\mathcal{C}_1^{\bar{n}_1}, \mathcal{C}_2^{\bar{n}_2}, \dots, \mathcal{C}_K^{\bar{n}_K}\}$ with each member $\mathcal{C}_i^{\bar{n}_i}$ having block length $\bar{n}_i \triangleq \sum_{j \in [i]} n_j$, rate R_i , and a generator matrix $\mathbf{G}_i = [\mathbf{S}_1 \mathbf{S}_2 \dots \mathbf{S}_i]$, since $\{\mathbf{G}_i\}_{i \in [K]}$ forms a sequence of nested generating matrices, i.e. $\mathbf{G}_i \subseteq \mathbf{G}_{i+1}$ for all $i \in [K-1]$. The K -level PCP code with generator matrix \mathbf{G}_K may be viewed as the "mother" code of the lowest rate in the rate compatible family.

B. Sequential Decoding

To decode the constructed parallel concatenated polar code we use sequential decoder shown in Algorithm 1. In the decoding given by Algorithm 1, information bits are decoded by K polar decoders sequentially. The number of bit decisions made by the sequential decoder is independent of the number of decoding stages K and it equals the number of information bits k . The sequential decoder approach is optimal in the sense that it achieves the capacity as all block lengths approach infinity. Since reception from previous retransmissions contains

useful information, the finite-block length performance can

Algorithm 1 Decoding algorithm

```

1: procedure DECODER( $\mathbf{y}^{\bar{n}_K}$ ) ▷ Input: received vector  $\mathbf{y}^{\bar{n}_K}$ 
2:   Decode  $n_K R_K$  bits  $\mathcal{I}_K^{(K)}$  using  $\mathcal{C}(n_K, R_K, \mathcal{A}_K^{(K)})$ 
3:   for  $i = K-1, \dots, 1$  do
4:     Use  $\bigcup_{j=i}^{K-1} \mathcal{I}_j^{(i)}$  as frozen bits in  $\mathcal{C}(n_i, R_i, \mathcal{A}_i^{(i)})$  to
       get  $\mathcal{C}(n_i, R_K, \mathcal{A}_K^{(i)})$ 
5:     Decode  $n_i R_K$  bits  $\mathcal{I}_K^{(i)}$  using  $\mathcal{C}(n_i, R_K, \mathcal{A}_K^{(i)})$ 
6:   end for
7:   return  $\mathcal{I}^{(1)} = \bigcup_{i=1}^K \mathcal{I}_K^{(i)}$  ▷ Output: decoded bits  $\mathcal{I}^{(1)}$ 
8: end procedure

```

be improved by employing a soft decoder; each component code receives a soft information from other component codes and exploits it as a priori information as performed in Turbo code. The study of an enhanced decoder is left for a future work. Simulation results for the finite block length performance can be found in [12].

VI. CONCLUSION

A method of constructing rate-compatible polar codes that are capacity-achieving with low-complexity sequential decoders is presented. Due to the length limitation of polar codes, the proposed construction cannot support an arbitrary sequence of rates, and we characterize the rates that can be supported. We then present capacity-achieving punctured polar codes that provide more flexibility on block length by controlling a puncturing fraction. We finally show that by using such punctured polar codes, the proposed rate-compatible polar code is capacity-achieving for an arbitrary sequence of rates and for any class of degraded channels.

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